

Quantum Aharonov-Bohm Billiard System

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(February 5, 2008)

Abstract

The Green's functions of the two and three-dimensional relativistic Aharonov-Bohm (A-B) systems are given by the path integral approach. In addition the exact radial Green's functions of the spherical A-B quantum billiard system in two and three-dimensional are obtained via the perturbation technique of δ -function.

PACS: 03.65.Bz; 03.65.Ge

Typeset using REVTeX

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I. INTRODUCTION

It was points out in 1959 by A-B [1] that the motion of a charged particle can be affected by magnetic fields in regions from which the particle is excluded. This interesting property of magnetic fields and thus the vector potential $\mathbf{A}(\mathbf{x})$ in quantum mechanics has been well confirmed experimentally. Because it raises some fundamental questions, the A-B effect has much debated and written about over the years [2]. Recently, the A-B billiard systems have been much interest in the contexts of mesoscope [3–7], nonlinear, and semi-classical dynamics [8,9].

In this contribution, we present the Green's functions of the relativistic 2 and 3-dimensional A-B system by the path integral approach. Furthermore, the exact Green's functions of 2 and 3-dimensional relativistic spherical quantum billiard with a singular magnetic flux (A-B magnetic field) centered at the origin is given by the closed formula of the perturbation technique. It is found that the energy spectra is only determined by the modified Bessel function involving the partial wave Green's function of the unperturbed relativistic A-B effect.

II. GREEN'S FUNCTION OF THE TWO-DIMENSIONAL A-B BILLIARD SYSTEM

The starting point is the path integral representation for the Green's function of a relativistic particle in external electromagnetic fields [10,11]:

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dL \int \mathcal{D}\rho(\lambda) \Phi[\rho(\lambda)] \int \mathcal{D}^D x(\lambda) \exp \{ -A_E[\mathbf{x}, \dot{\mathbf{x}}] / \hbar \} \rho(0) \quad (2.1)$$

with the action

$$A_E[\mathbf{x}, \dot{\mathbf{x}}] = \int_{\lambda_a}^{\lambda_b} d\lambda \left[\frac{m}{2\rho(\lambda)} \dot{\mathbf{x}}^2(\lambda) - i(e/c) \mathbf{A}(\mathbf{x}) \cdot \dot{\mathbf{x}}(\lambda) - \rho(\lambda) \frac{(E - V(\mathbf{x}))^2}{2mc^2} + \rho(\lambda) \frac{mc^2}{2} \right], \quad (2.2)$$

where L is defined as

$$L = \int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda), \quad (2.3)$$

in which $\rho(\lambda)$ is an arbitrary dimensionless fluctuating scale variable, $\rho(0)$ is the terminal point of the function $\rho(\lambda)$, and $\Phi[\rho(\lambda)]$ is some convenient gauge-fixing functional [10–12]. The only condition on $\Phi[\rho(\lambda)]$ is that

$$\int \mathcal{D}\rho(\lambda) \Phi[\rho(\lambda)] = 1. \quad (2.4)$$

\hbar/mc is the well-known Compton wave length of a particle of mass m , $\mathbf{A}(\mathbf{x})$ and $V(\mathbf{x})$ stand for the vector and scalar potential of the systems, respectively. E is the system energy, and \mathbf{x} is the spatial part of the $(D+1)$ vector $x^\mu = (\mathbf{x}, \tau)$.

For the pure Aharonov-Bohm (AB) system under consideration, the scalar potential $V(\mathbf{x}) = 0$ and the vector potential reads

$$\mathbf{A}(\mathbf{x}) = 2g \frac{-y\hat{e}_1 + x\hat{e}_2}{x^2 + y^2}, \quad (2.5)$$

where $\hat{e}_{1,2}$ stand for the unit vector along the x, y axis, respectively. In this problem, the functional $\Phi[\rho(\lambda)]$ can be taken as the δ -functional $\delta[\rho - 1]$ to fixed the value of $\rho(\lambda)$ to unity [11]. For convenience, Let's now introduce the azimuthal angle around the magnetic tube:

$$\varphi(\mathbf{x}) = \arctan(y/x). \quad (2.6)$$

The components of the vector potential can be therefore expressed as

$$A_i = 2g \partial_i \varphi(\mathbf{x}). \quad (2.7)$$

The associated magnetic field lines are confined to an infinitely thin tube along the z-axis:

$$B_3 = 2g \epsilon_{3ij} \partial_i \partial_j \varphi(\mathbf{x}) = 4\pi g \delta(\mathbf{x}_\perp), \quad (2.8)$$

where \mathbf{x}_\perp stands for the transverse vector $\mathbf{x}_\perp = (x, y)$. Note that the derivatives in front of $\varphi(\mathbf{x})$ commute everywhere, except at the origin where Stokes' theorem yields

$$\int d^2x (\partial_1 \partial_2 - \partial_2 \partial_1) \varphi(\mathbf{x}) = \oint d\varphi = 2\pi. \quad (2.9)$$

The magnetic flux through the tube is defined by the integral

$$\Omega = \int d^2x B_3. \quad (2.10)$$

This shows that the coupling constant g is related to the magnetic flux by

$$g = \frac{\Omega}{4\pi}. \quad (2.11)$$

Inserting $A_i = 2g\partial_i\varphi(\mathbf{x})$ in Eq. (2.2), the magnetic interaction takes the form

$$A_{\text{mag}} = -\hbar\beta_0 \int_0^L d\lambda \dot{\varphi}(\lambda), \quad (2.12)$$

where $\varphi(\lambda) = \varphi(\mathbf{x}(\lambda))$, and β_0 is the dimensionless number

$$\beta_0 = -\frac{2eg}{\hbar c}. \quad (2.13)$$

The minus sign is a matter of convention. Since the particle orbits are present at all times, their worldlines in spacetime can be considered as being closed at infinity, and the integral

$$n = \frac{1}{2\pi} \int_0^L d\lambda \dot{\varphi}(\lambda) \quad (2.14)$$

is the topological invariant with integer values of the winding number n . The magnetic interaction is therefore purely topological, its value being

$$A_{\text{mag}} = -\hbar\beta_0 2n\pi. \quad (2.15)$$

After adding this to the action of Eq. (2.2) in the radial decomposition of the relativistic path integral, we rewrite the sum over the azimuthal quantum number via Poisson's summation formula

$$\sum_{k=-\infty}^{\infty} f(k) = \int_{-\infty}^{\infty} dy \sum_{n=-\infty}^{\infty} e^{2\pi n y i} f(y). \quad (2.16)$$

This leads to

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dL e^{L\mathcal{E}/\hbar} \int_{-\infty}^\infty d\alpha K(r_b, r_a; L)_\alpha \cdot \sum_{n=-\infty}^\infty \frac{1}{2\pi} e^{i(\alpha-\beta_0)(\varphi_b+2n\pi-\varphi_a)}, \quad (2.17)$$

where the pseudoenergy $\mathcal{E} = (E^2 - m^2 c^4) / 2mc^2$, and the radial pseudopropagator $K(r_b, r_a; L)_\alpha$ has the representation

$$K(r_b, r_a; L)_\alpha = \frac{m}{\hbar} \frac{1}{L} e^{-m(r_b^2 + r_a^2)/2\hbar L} I_\alpha\left(\frac{mr_b r_a}{\hbar L}\right). \quad (2.18)$$

The sum over all n in Eq. (2.17) forces α to be equal to β_0 modulo an arbitrary integral number. The result is

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dL e^{L\mathcal{E}/\hbar} K(\mathbf{x}_b, \mathbf{x}_a; L) \quad (2.19)$$

in which $K(\mathbf{x}_b, \mathbf{x}_a; L)$ is given by

$$K(\mathbf{x}_b, \mathbf{x}_a; L) = \sum_{n=-\infty}^\infty K(r_b, r_a; L)_{n+\beta_0} \frac{1}{2\pi} e^{in(\varphi_b - \varphi_a)}. \quad (2.20)$$

From Eq. (2.19), we observe that $K(\mathbf{x}_b, \mathbf{x}_a; L)$ can be viewed as the propagator of AB system with the pseudoenergy \mathcal{E} . The entire Green's function can be obtained by doing the integration. At this place, let us first discuss the wave function properties by noting the relation

$$\Psi(r, \varphi; L) = \int_0^\infty r_0 dr_0 \int_{-\pi}^\pi d\varphi_0 K(\mathbf{x}, \mathbf{x}_0; L) \Psi(r_0, \varphi_0; 0), \quad (2.21)$$

where $\Psi(r_0, \varphi_0; 0)$ reads

$$\Psi(r_0, \varphi_0; 0) = e^{-ikr_0 \cos \varphi_0} e^{-i\beta_0 \varphi_0} \quad (2.22)$$

with the $k = \sqrt{E^2 - m^2 c^4} / \hbar c$. Inserting the expression of Eq. (2.20) in Eq. (2.21), we obtain

$$\begin{aligned} \Psi(r, \varphi; L) &= \sum_{n=-\infty}^\infty e^{in\varphi} \int_0^\infty r_0 dr_0 \frac{m}{\hbar} \frac{1}{L} e^{-m(r^2 + r_0^2)/2\hbar L} I_{n+\beta_0}\left(\frac{mrr_0}{\hbar L}\right) \\ &\times \frac{1}{2\pi} \int_{-\pi}^\pi d\varphi_0 e^{-ikr_0 \cos \varphi_0 - i(n+\beta_0)\varphi_0}. \end{aligned} \quad (2.23)$$

The angular integration can be performed by using the formula

$$\lim_{z \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-iz \cos \theta - i\nu\theta} = I_{\nu}(-iz). \quad (2.24)$$

We arrive at

$$\Psi(r, \varphi; L) = \sum_{n=-\infty}^{\infty} e^{in\varphi} \int_0^{\infty} r_0 dr_0 \frac{m}{\hbar} \frac{1}{L} e^{-m(r^2+r_0^2)/2\hbar L} I_{n+\beta_0} \left(\frac{mrr_0}{\hbar L} \right) I_{n+\beta_0}(-ikr_0). \quad (2.25)$$

The integral can perform with the help of the formula

$$\int_0^{\infty} dr r e^{-r^2/a} I_{\nu}(\varsigma r) I_{\nu}(\xi r) = \frac{a}{2} e^{a(\xi^2+\varsigma^2)/4} I_{\nu} \left(\frac{a\xi\varsigma}{2} \right). \quad (2.26)$$

It yields

$$\Psi(r, \varphi; L) = \sum_{n=-\infty}^{\infty} e^{in\varphi} I_{n+\beta_0}(-ikr) e^{-(\hbar k)^2 L/2m\hbar}. \quad (2.27)$$

Extracting the wave function depending only on spatial variable (r, φ) , we obtain

$$\Psi(r, \varphi) = \sum_{n=-\infty}^{\infty} (-i)^{n+\beta_0} J_{n+\beta_0}(kr) e^{in\varphi}, \quad (2.28)$$

where the equality $I_{\nu}(-iz) = (-i)^{\nu} J_{\nu}(z)$ has been used. This wave function coincides with the local approach in the relativistic version. The scattering wave of the relativistic AB effect can be extracted by splitting the wave function into three parts:

$$\begin{aligned} \Psi &= \Psi_1 + \Psi_2 + \Psi_3 \\ &= \sum_{n=1}^{\infty} (-i)^{n+\beta_0} J_{n+\beta_0}(kr) e^{in\varphi} + \sum_{n=1}^{\infty} (-i)^{n-\beta_0} J_{n-\beta_0}(kr) e^{-in\varphi} + (-i)^{\beta_0} J_{\beta_0}(kr) e^{in\varphi}. \end{aligned} \quad (2.29)$$

It is not difficult to calculate the asymptotic behavior is given by

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 \xrightarrow{r \rightarrow \infty} e^{-ikr \cos \varphi} e^{-i\beta_0 \varphi} + f(\varphi) \frac{e^{ikr}}{\sqrt{kr}} \quad (2.30)$$

with the scattering amplitude

$$f(\varphi) = \frac{e^{-i3\pi/4}}{\sqrt{2\pi}} \sin \pi \beta_0 \frac{e^{-i\varphi/2}}{\cos(\varphi/2)}. \quad (2.31)$$

The corresponding cross section is

$$\frac{d\sigma}{d\varphi} = \frac{\sin^2(\pi\beta_0)}{2\pi} \frac{1}{\cos^2(\varphi/2)}. \quad (2.32)$$

It has a strong peak near the forward direction $\varphi \approx \pi$. For $\beta_0 = \text{integer}$, there is no scattering at all. This concludes the discussion of the wave function by the path integral solution of the relativistic spinless AB system.

Now let's begin to discuss the A-B billiard system. With a method developed in Ref. [13,14], the exact Green's function of quantum billiard for a spherically shaped, impenetrable wall located at the radius $r = a$ is given by the following formula

$$G^{(\text{wall})}(r_b, r_a; E) = G(r_b, r_a; E) - \frac{G(r_b, a; E)G(a, r_a; E)}{G(a, a; E)}, \quad (2.33)$$

where $G(r_b, r_a; E)$ is the Green's function of unperturbed radial propagator. For the pure A-B system under consideration, It can be obtained by noting Eqs. (2.19) and (2.20) and reads

$$G(r_b, r_a; E) = \int_0^\infty dL e^{L\mathcal{E}/\hbar} \frac{m}{\hbar} \frac{1}{L} e^{-m(r_b^2 + r_a^2)/2\hbar L} I_{|n+\beta_0|} \left(\frac{mr_b r_a}{\hbar L} \right). \quad (2.34)$$

The integral can perform by using the integral representation [[15], p.200],

$$\int_0^\infty \frac{dz}{z} e^{-pz - (a+b)/2z} I_\nu \left(\frac{a-b}{2z} \right) = 2I_\nu \left(\sqrt{p}(\sqrt{a} - \sqrt{b}) \right) K_\nu \left(\sqrt{p}(\sqrt{a} + \sqrt{b}) \right), \quad (2.35)$$

yielding

$$\begin{aligned} G(r_b, r_a; E) &= \frac{2m}{\hbar} I_{|n+\beta_0|} \left(\sqrt{-m\mathcal{E}/2\hbar^2} (r_b + r_a - |r_b - r_a|) \right) \\ &\times K_{|n+\beta_0|} \left(\sqrt{-m\mathcal{E}/2\hbar^2} (r_b + r_a + |r_b - r_a|) \right). \end{aligned} \quad (2.36)$$

This gives the Green's function with a wall located at $r = a$ e.g. for $r_a < r_b < a$:

$$\begin{aligned} G^{(\text{wall})}(r_b, r_a; E) &= \frac{2m}{\hbar} \left[I_{|n+\beta_0|} \left(\sqrt{-2m\mathcal{E}/\hbar^2} a \right) K_{|n+\beta_0|} \left(\sqrt{-2m\mathcal{E}/\hbar^2} r_b \right) - (K \leftrightarrow I) \right] \\ &\times \frac{I_{|n+\beta_0|} \left(\sqrt{-2m\mathcal{E}/\hbar^2} r_a \right)}{I_{|n+\beta_0|} \left(\sqrt{-2m\mathcal{E}/\hbar^2} a \right)}. \end{aligned} \quad (2.37)$$

The corresponding bound state energy spectra are given by the equation

$$I_{|n+\beta_0|} \left(\sqrt{-2m\mathcal{E}/\hbar^2 a} \right) = 0. \quad (2.38)$$

We see that the presence of the flux line in the circular billiard simply changes the order of the Bessel functions from the integer to fractional. If we assume that $-\beta_0$ can take a continuous range of values between 0 and 1, the symmetry $|\beta_0| \leftrightarrow (1 + |\beta_0|)$ in the quantum spectrum allows the restriction to $0 \leq |\beta_0| \leq 0.5$. For integer flux $|\beta_0| = 0, 1, 2, \dots$, the quantum spectrum is unaltered by the flux line. This is seen from the fact that for any integer value of β_0 the angular momentum gets redefined and the new set is isomorphic to the old one both in terms of the spectrum and eigenstates. This mapping, however, has no classical analog since the classically allowed angular momenta remain the same.

III. GREEN'S FUNCTION OF THE THREE-DIMENSIONAL A-B BILLIARD SYSTEM

For solving the path integral of A-B system in three-dimensional space, Let's now introduce the space-time transformation [16,17]

$$\epsilon_n^\lambda = \epsilon_n^s f(\mathbf{x}_n) \quad (3.1)$$

with the short “time” interval $\epsilon_n^\lambda = \lambda_n - \lambda_{n-1}$ to regularize the path integral in time-sliced form for getting a tractable one. Then we have the regularization path integral as following:

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2mc} \int_0^\infty dS \frac{f(\mathbf{x}_a)}{\left(\frac{2\pi\hbar\epsilon_b^s f(\mathbf{x}_b)}{m} \right)^{3/2}} \prod_{n=1}^N \left[\int_{-\infty}^\infty \frac{d^3 x_n}{\left(\frac{2\pi\hbar\epsilon_n^s f(\mathbf{x}_n)}{m} \right)^{3/2}} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (3.2)$$

with the s -sliced action

$$A^N = \sum_{n=1}^{N+1} \left[\frac{m(\mathbf{x}_n - \mathbf{x}_{n-1})^2}{2\epsilon_n^s f(\mathbf{x}_n)} - i\frac{e}{c} \mathbf{A}(\mathbf{x}_n) \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \epsilon_n^s f(\mathbf{x}_n) \frac{E^2}{2mc^2} + \epsilon_n^s f(\mathbf{x}_n) \frac{mc^2}{2} \right]. \quad (3.3)$$

The sign \approx in Eq. (3.2) becomes an equality for $N \rightarrow \infty$. The regularization $f(\mathbf{x})$ in this problem can be chosen as the radial distance $r = \sqrt{x^2 + y^2 + z^2}$ and thus $f(\mathbf{x}_a) = r_a$. Since

we shall introduce the KS-transformation [16], let us insert in Eq. (3.2) a functional integral representation of unity

$$\prod_{n=1}^{N+1} \left[\int \frac{d\Delta w_n}{(2\pi\hbar\epsilon_n^s\rho_n r_n/m)^{1/2}} \right] \exp \left\{ -\frac{1}{\hbar} \sum_{n=1}^{N+1} \frac{m}{2} \frac{(\Delta w_n)^2}{\epsilon_n^s\rho_n r_n} \right\} = 1. \quad (3.4)$$

The w_n is a fictitious fourth coordinate axis. With this, the path integral of 3-dimensional A-B effect can be rewritten as the 4-dimensional path integral in the time-sliced version

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2mc} \int_0^\infty dS \int dw_a \frac{r_a}{\left(\frac{2\pi\hbar\epsilon_b^s r_b}{m}\right)^2} \prod_{n=2}^{N+1} \left[\int_{-\infty}^\infty \frac{d\Delta \vec{x}_n}{\left(\frac{2\pi\hbar\epsilon_n^s r_{n-1}}{m}\right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (3.5)$$

with the sliced action

$$A^N = \sum_{n=1}^{N+1} \left[\frac{m(\vec{x}_n - \vec{x}_{n-1})^2}{2\epsilon_n^s r_n} - i(e/c) \mathbf{A}(\mathbf{x}_n) \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \epsilon_n^s r_n \frac{E^2}{2mc^2} + \epsilon_n^s r_n \frac{mc^2}{2} \right], \quad (3.6)$$

where the kinetic term in Eq. (3.3) has been replaced with the four-vector \vec{x} due to the reason of Eq. (3.4) and the integrals over $\Delta \vec{x}_n$ may be performed successively from $n = N$ down to $n = 1$. The notation's change of the measure of integration is necessary for discussing the path integral in curved space, since \vec{x}_n are Cartesian coordinates and certainly identical in the time-sliced expressions [17]:

$$\prod_{n=1}^N \left[\int_{-\infty}^\infty d\vec{x}_n \right] = \prod_{n=2}^{N+1} \left[\int_{-\infty}^\infty d\Delta \vec{x}_n \right]. \quad (3.7)$$

However, their images under a non-holonomic mapping are different so that the initial form of the sliced path integral is a matter of choice. In the space with curvature and torsion it has been proved in Ref. [17]. Only the right-hand side of Eq. (3.7) gives the properly correct results. To go further, let's adjust for having the same time-sliced index the measure via the following approximation

$$\frac{r_a}{\left(\frac{2\pi\hbar\epsilon_b^s r_b}{m}\right)^2} \prod_{n=2}^{N+1} \left[\int_{-\infty}^\infty \frac{d\Delta \vec{x}_n}{\left(\frac{2\pi\hbar\epsilon_n^s r_{n-1}}{m}\right)^2} \right] \approx \frac{1}{r_a} \frac{1}{\left(\frac{2\pi\hbar\epsilon_b^s}{m}\right)^2} \prod_{n=2}^{N+1} \left[\int_{-\infty}^\infty \frac{d\Delta \vec{x}_n}{\left(\frac{2\pi\hbar\epsilon_n^s r_n}{m}\right)^2} \right]. \quad (3.8)$$

With the help of the approximation, we arrive at

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2mc} \int_0^\infty dS \int \frac{dw_a}{r_a} \frac{1}{\left(\frac{2\pi\hbar\epsilon_b^s}{m}\right)^2} \prod_{n=2}^{N+1} \left[\int_{-\infty}^\infty \frac{d\Delta \vec{x}_n}{\left(\frac{2\pi\hbar\epsilon_n^s r_n}{m}\right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\}. \quad (3.9)$$

This path integral can be simplified by the KS-transformation

$$d\vec{x} = 2A(\vec{u})d\vec{u}. \quad (3.10)$$

The 4×4 matrix $A(\vec{u})$ is chosen as

$$A(\vec{u}) = \begin{pmatrix} u^3 & u^4 & u^1 & u^2 \\ u^4 & -u^3 & -u^2 & u^1 \\ u^1 & u^2 & -u^3 & -u^4 \\ u^2 & -u^1 & u^4 & -u^3 \end{pmatrix}. \quad (3.11)$$

The transformations of the volume element and velocity are given as

$$d\vec{x} = 16r^2 d\vec{u}, \quad (3.12)$$

$$\vec{x}'^2 = 4\vec{u}'^2 \vec{u}^2 = 4r\vec{u}'^2. \quad (3.13)$$

Furthermore, the magnetic interaction under the KS-transformation turns into

$$\mathbf{A}(\mathbf{x}) \cdot \dot{\mathbf{x}}(\lambda) = -2g \frac{y\dot{x} - x\dot{y}}{x^2 + y^2} = -2g \left[\frac{u^1 \dot{u}^2 - u^2 \dot{u}^1}{(u^1)^2 + (u^2)^2} + \frac{u^4 \dot{u}^3 - u^3 \dot{u}^4}{(u^3)^2 + (u^4)^2} \right] \quad (3.14)$$

or in the time-sliced version

$$\begin{aligned} \mathbf{A}(\mathbf{x}_n) \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) &= -2g \frac{y_n \Delta x_n - x_n \Delta y_n}{x_n^2 + y_n^2} \\ &= -2g \left[\frac{u_n^1 \Delta u_n^2 - u_n^2 \Delta u_n^1}{(u_n^1)^2 + (u_n^2)^2} + \frac{u_n^4 \Delta u_n^3 - u_n^3 \Delta u_n^4}{(u_n^3)^2 + (u_n^4)^2} \right]. \end{aligned} \quad (3.15)$$

We obtain a path integral equivalent to Eq. (3.9)

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dS G(\vec{u}_b, \vec{u}_a; S), \quad (3.16)$$

where $G(\vec{u}_b, \vec{u}_a; S)$ denotes the s-sliced amplitude

$$G(\vec{u}_b, \vec{u}_a; S) \approx \frac{1}{16} \int \frac{dw_a}{r_a} \frac{1}{\left(\frac{2\pi\hbar\epsilon_a^s}{M}\right)^2} \prod_{n=1}^N \left[\int_{-\infty}^\infty \frac{d\vec{u}_n}{\left(\frac{2\pi\hbar\epsilon_n^s}{M}\right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (3.17)$$

with the action

$$A^N = \sum_{n=1}^{N+1} \left\{ \frac{M(\Delta \vec{u}_n)^2}{2\epsilon_n^s} - i(e/c) [\vec{A}(u_n) \cdot \Delta \vec{u}_n] + \epsilon_n^s \frac{M\omega^2 \vec{u}_n^2}{2} \right\}. \quad (3.18)$$

Here

$$M = 4m, \quad \omega^2 = \frac{m^2 c^4 - E^2}{4m^2 c^2}, \quad (3.19)$$

and

$$\vec{A}(u_n) \cdot \Delta \vec{u}_n = -2g \left[\frac{u_n^1 \Delta u_n^2 - u_n^2 \Delta u_n^1}{(u_n^1)^2 + (u_n^2)^2} + \frac{u_n^4 \Delta u_n^3 - u_n^3 \Delta u_n^4}{(u_n^3)^2 + (u_n^4)^2} \right]. \quad (3.20)$$

In the continuum limit, this amounts to

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dS \frac{1}{16} \int \frac{dw_a}{r_a} \int \mathcal{D}\vec{u}(s) \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (3.21)$$

with the action

$$A = \int_0^S ds \left[\frac{M\vec{u}'^2}{2} - 2i(e/c)(\vec{A}(s) \cdot \vec{u}'(s)) + \frac{M\omega^2 \vec{u}^2}{2} \right]. \quad (3.22)$$

There are no s -slicing corrections. This is ensured by the affine connection of KS-transformation satisfying

$$\Gamma_\mu^{\mu\lambda} = g^{\mu\nu} e_i^\lambda \partial_\mu e_\nu^i = 0 \quad (3.23)$$

with the basis triads $e_\nu^i(q) = \partial x^i / \partial q^\nu$ and the transverse gauge $\partial_\mu A^\mu = 0$ [17]. Note that the system becomes separable like $R^4 \rightarrow R^2 \times R^2$ in which each R^2 has a dynamical model a 2-dimensional simple harmonic oscillator moving in the A-B magnetic fields. Its Green's function is given by [16,17]

$$\frac{M\omega}{\hbar \sinh \omega s} \sum_{k=-\infty}^{\infty} e^{ik(\varphi_b - \varphi_a)} \exp \left\{ -\frac{M\omega}{2\hbar} (\sigma_b^2 + \sigma_a^2) \coth \omega s \right\} I_{|k+\beta_0|} \left(\frac{M}{\hbar} \frac{\omega \sigma_b \sigma_a}{\sinh \omega s} \right), \quad (3.24)$$

where $\sigma = \sqrt{x^2 + y^2}$ is the radial length, I_ν is the modified Bessel function, and $\beta_0 \equiv -2eg/\hbar c$. Therefore, we obtain the entire Green's function

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dS \frac{1}{16} \int \frac{dw_a}{r_a} \left(\frac{M\omega}{\hbar \sinh \omega s} \right)^2$$

$$\begin{aligned}
& \times \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{ik_1(\varphi_{1,b}-\varphi_{1,a})} e^{ik_2(\varphi_{2,b}-\varphi_{2,a})} \\
& \times \exp \left\{ -\frac{M\omega}{2\hbar} \left(\sigma_{1,b}^2 + \sigma_{1,a}^2 + \sigma_{2,b}^2 + \sigma_{2,a}^2 \right) \coth \omega s \right\} \\
& \times I_{|k_1+\beta_0|} \left(\frac{M}{\hbar} \frac{\omega \sigma_{1,b} \sigma_{1,a}}{\sinh \omega s} \right) I_{|k_2+\beta_0|} \left(\frac{M}{\hbar} \frac{\omega \sigma_{2,b} \sigma_{2,a}}{\sinh \omega s} \right). \tag{3.25}
\end{aligned}$$

Here the coordinate transformations (σ_1, φ_1) and (σ_2, φ_2) are defined as

$$\left. \begin{aligned} u^1 &= \sigma_1 \sin \varphi_1 \\ u^2 &= \sigma_1 \cos \varphi_1 \\ u^3 &= \sigma_2 \cos \varphi_2 \\ u^4 &= \sigma_2 \sin \varphi_2 \end{aligned} \right\}. \tag{3.26}$$

To perform the w_a integration, let's express the variables $(\sigma_1, \varphi_1, \sigma_2, \varphi_2)$ in terms of the Euler angle variables by defining:

$$\left. \begin{aligned} u^1 &= \sqrt{r} \cos(\theta/2) \cos[(\varphi + \gamma)/2] \\ u^2 &= -\sqrt{r} \cos(\theta/2) \sin[(\varphi + \gamma)/2] \\ u^3 &= \sqrt{r} \sin(\theta/2) \cos[(\varphi - \gamma)/2] \\ u^4 &= \sqrt{r} \sin(\theta/2) \sin[(\varphi - \gamma)/2] \end{aligned} \right\} \begin{pmatrix} 0 \leq \theta \leq \pi \\ 0 \leq \varphi \leq 2\pi \\ 0 \leq \gamma \leq 4\pi \end{pmatrix} \tag{3.27}$$

and identify

$$\left. \begin{aligned} \sigma_1 &= \sqrt{r} \cos(\theta/2) \\ \varphi_1 &= (\varphi + \gamma + \pi)/2 \\ \sigma_2 &= \sqrt{r} \sin(\theta/2) \\ \varphi_2 &= (\varphi - \gamma)/2 \end{aligned} \right\}. \tag{3.28}$$

Then one can change the w_a -integration into the γ_a -integration whose result is easily represented as the Kronecker delta δ_{k_1, k_2} . Hence, we carry out k_2 -summation and finally becomes

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2mc} \frac{m^2 \omega}{\pi \hbar^2} \sum_{k=-\infty}^{\infty} e^{ik(\varphi_b - \varphi_a)}$$

$$\begin{aligned}
& \times \int_0^\infty d\eta \frac{1}{\sinh^2 \eta} e^{-\frac{M\omega}{2\hbar}(r_b+r_a) \coth \eta} \\
& \times I_{|k+\beta_0|} \left(\frac{M\omega \sqrt{r_b r_a}}{\hbar \sinh \eta} \cos \theta_b / 2 \cos \theta_a / 2 \right) I_{|k+\beta_0|} \left(\frac{M\omega \sqrt{r_b r_a}}{\hbar \sinh \eta} \sin \theta_b / 2 \sin \theta_a / 2 \right), \quad (3.29)
\end{aligned}$$

where we have defined the new variable $\eta = \omega s$. The product of modified Bessel functions can simplify by making use of the addition theorem [11]

$$\begin{aligned}
& I_\nu(z \sin \alpha / 2 \sin \beta / 2) I_\mu(z \cos \alpha / 2 \cos \beta / 2) \\
& = \frac{2}{z} (\sin \alpha / 2 \sin \beta / 2)^\nu (\cos \alpha / 2 \cos \beta / 2)^\mu \sum_{l=0}^{\infty} \frac{l! \Gamma(l + \mu + \nu + 1) (2l + \mu + \nu + 1)}{\Gamma(l + \mu + 1) \Gamma(l + \nu + 1)} \\
& \times I_{2l+\mu+\nu+1}(z) P_l^{(\mu, \nu)}(\cos \theta_b) P_l^{(\mu, \nu)}(\cos \theta_a), \quad (3.30)
\end{aligned}$$

where $P_l^{(\mu, \nu)}$ is Jacobi polynomial (e.g. p.209 [18]). The Green's function in Eq. (3.29) becomes

$$\begin{aligned}
G(\mathbf{x}_b, \mathbf{x}_a; E) &= \frac{i\hbar}{2mc} \frac{m}{2\pi\hbar\sqrt{r_b r_a}} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} e^{ik(\varphi_b - \varphi_a)} \\
& \times (\cos \theta_b / 2 \cos \theta_a / 2 \sin \theta_b / 2 \sin \theta_a / 2)^{|k+\beta_0|} \frac{l! \Gamma(l + 2 + |k + \beta_0| + 1) (2l + 2 + |k + \beta_0| + 1)}{\Gamma^2(l + |k + \beta_0| + 1)} \\
& \times \left\{ \int_0^\infty d\eta \frac{1}{\sinh \eta} e^{-\frac{M\omega}{2\hbar}(r_b+r_a) \coth \eta} I_{2l+2|k+\beta_0|+1} \left(\frac{M\omega \sqrt{r_b r_a}}{\hbar \sinh \eta} \right) \right\} \\
& \times P_l^{(|k+\beta_0|, |k+\beta_0|)}(\cos \theta_b) P_l^{(|k+\beta_0|, |k+\beta_0|)}(\cos \theta_a). \quad (3.31)
\end{aligned}$$

The integral can perform by noting the equality [12]

$$\begin{aligned}
& \int_0^\infty dz \frac{1}{\sinh z} e^{-\frac{M\omega}{2\hbar}(r_b+r_a) \coth z} I_\nu \left(\frac{M\omega \sqrt{r_b r_a}}{\hbar \sinh z} \right) \\
& = \frac{1}{2} \int_0^\infty \frac{dS}{S} e^{-\frac{\varepsilon}{\hbar} S} e^{-m(r_b^2 + r_a^2)/2\hbar S} I_{\nu/2} \left(\frac{m r_b r_a}{\hbar S} \right), \quad (3.32)
\end{aligned}$$

where \mathcal{E} is defined as $(m^2 c^4 - E^2)/2mc^2$. We finally obtain the exact Green's function of the relativistic three-dimensional A-B effect

$$\begin{aligned}
G(\mathbf{x}_b, \mathbf{x}_a; E) = & \frac{i\hbar}{2mc} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} e^{ik(\varphi_b - \varphi_a)} (\cos \theta_b / 2 \cos \theta_a / 2 \sin \theta_b / 2 \sin \theta_a / 2)^{|k+\beta_0|} \\
& \times \frac{m}{2\pi\hbar\sqrt{r_b r_a}} \frac{l! \Gamma(l+2 | k+\beta_0 | +1) (2l+2 | k+\beta_0 | +1)}{\Gamma^2(l+ | k+\beta_0 | +1)} \\
& \times \left\{ I_\nu \left[\sqrt{\frac{m\mathcal{E}}{2\hbar^2}} ((r_b + r_a) - |r_b - r_a|) \right] K_\nu \left[\sqrt{\frac{m\mathcal{E}}{2\hbar^2}} ((r_b + r_a) + |r_b - r_a|) \right] \right\} \\
& \times P_l^{(|k+\beta_0|, |k+\beta_0|)} (\cos \theta_b) P_l^{(|k+\beta_0|, |k+\beta_0|)} (\cos \theta_a)
\end{aligned} \tag{3.33}$$

with $\nu = l + |k + \beta_0| + 1/2$. It is worth to note that there exist no bound states in the pure A-B effect. This is reasonable, since we are treating a scattering system.

According to the orthogonality relations of Jacobi polynomials [18],

$$\begin{aligned}
& \int_{-1}^{-1} dx (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) \\
& = \frac{2^{\alpha+\beta+1}}{\alpha + \beta + 2n + 1} \frac{\Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{n! \Gamma(\alpha + \beta + n + 1)} \delta_{m,n}
\end{aligned} \tag{3.34}$$

,we find the radial Green's function of the relativistic three-dimensional A-B effect

$$\begin{aligned}
G(r_b, r_a; E) = & \frac{2m}{\hbar\sqrt{r_b r_a}} \\
& \times \left\{ I_\nu \left[\sqrt{\frac{m\mathcal{E}}{2\hbar^2}} ((r_b + r_a) - |r_b - r_a|) \right] K_\nu \left[\sqrt{\frac{m\mathcal{E}}{2\hbar^2}} ((r_b + r_a) + |r_b - r_a|) \right] \right\}.
\end{aligned} \tag{3.35}$$

By applying the method developed in Refs. [13,14] again, the effect of a spherically shaped impenetrable wall located at the radius $r = a$ can be researched via formula of Eq. (2.33).

This gives ,e.g. for $r_a < r_b < a$, the exact Green's function:

$$G^{(\text{wall})}(r_b, r_a; E) = \frac{2m}{\hbar\sqrt{r_b r_a}}$$

$$\times \left[I_\nu \left(\sqrt{2m\mathcal{E}/\hbar^2} a \right) K_\nu \left(\sqrt{2m\mathcal{E}/\hbar^2} r_b \right) - (K \leftrightarrow I) \right] \frac{I_\nu \left(\sqrt{2m\mathcal{E}/\hbar^2} r_a \right)}{I_\nu \left(\sqrt{2m\mathcal{E}/\hbar^2} a \right)}. \quad (3.36)$$

The corresponding bound state energy spectra are given by the equation

$$I_{l+|k+\beta_0|+1/2} \left(\sqrt{2m\mathcal{E}/\hbar^2} a \right) = 0. \quad (3.37)$$

We again see that the presence of the flux line in the circular billiard simply changes the order of the Bessel functions. The energy spectra is determined by the zero points of the modified Bessel function. This quantum effect may detect by the experiment. It has much interested in the mesoscope systems [3–7]. For the non-relativistic quantum A-B billiard system, the exact Green's function is given by replacing the pseudoenergy \mathcal{E} with $-E$.

IV. CONCLUDING REMARKS

In this contribution, the Green's function of the relativistic two and three-dimensional A-B system is given by path integral approach. The results are separated into the angular and radial parts. From the radial parts, the Green's function of the relativistic two and three-dimensional quantum A-B billiard system are obtained via the closed formula of Dirichlet boundary condition given by the δ -function perturbation. The energy spectra are determined by the zeros of the modified Bessel function involving the partial wave expanded Green's function of the unperturbed A-B systems. The A-B system serve as the prototype of arbitrary systems bounded by the spherical Dirichlet boundary condition. There is an interesting effect [3–5] in mesoscope systems related to our results for the non-relativistic case. Such effect can be exactly described by the A-B magnetic field surrounded by a spherically shaped δ -function. Its Green's function is given by [13,14]

$$G^{(\delta)}(r_b, r_a; E) = G(r_b, r_a; E) - \frac{G(r_b, a; E)G(a, r_a; E)}{G(a, a; E) - \hbar/\alpha a^{(D-1)}} \quad (4.1)$$

with G being the radial Green's function without $\delta(r - a)$ -potential and α the interacting strength of δ -function. In two-dimensional case, with the result of Eq. (2.36), Eq. (4.1) yields for $r_b > a > r_a$

$$G^{(\delta)}(r_b, r_a; E) = -\frac{2m}{\alpha a} \frac{I_{|n+\beta_0|} \left(\sqrt{-2mE/\hbar^2} r_a \right) K_{|n+\beta_0|} \left(\sqrt{-2mE/\hbar^2} r_b \right)}{\frac{2m}{\hbar} I_{|n+\beta_0|} \left(\sqrt{-2mE/\hbar^2} a \right) K_{|n+\beta_0|} \left(\sqrt{-2mE/\hbar^2} a \right) - \frac{\hbar}{\alpha a}}. \quad (4.2)$$

Energy spectra E_n of bound states are determined by the equation

$$\frac{\hbar^2}{2m\alpha a} = I_{|n+\beta_0|} \left(\sqrt{-2mE_n/\hbar^2} a \right) K_{|n+\beta_0|} \left(\sqrt{-2mE_n/\hbar^2} a \right). \quad (4.3)$$

From the asymptotic behavior of $I_\alpha(\alpha z) K_\alpha(\alpha z)$ for $\alpha \rightarrow \infty$ [p. 378 [19]]

$$I_\alpha(\alpha z) K_\alpha(\alpha z) \approx \frac{1}{2\alpha\sqrt{1+z^2}}, \quad (4.4)$$

we obtain

$$\frac{\hbar^2}{m\alpha a} \approx \left(|n+\beta_0| + \frac{2m|E_n|a^2}{\hbar^2} \right)^{-1/2} < \frac{1}{|n+\beta_0|}. \quad (4.5)$$

This implies only finite bound states exist and the upper bound is given by

$$|n+\beta_0| < \frac{m\alpha a}{\hbar^2}. \quad (4.6)$$

We see that the A-B effect not just change the energy levels but change the number of bound states. On the other hand, the radius of a thin-walled cylinder also affects the number of energy levels. For the three-dimensional case, with the radial Green's function (3.35) of A-B effect, we have the Green's function of semi-transparent wall for $r_b > a > r_a$

$$G^{(\delta)}(r_b, r_a; E) = -\frac{1}{\sqrt{r_b r_a}} \frac{\frac{2m}{\alpha a^2} I_{l+|k+\beta_0|+1/2} \left(\sqrt{-2mE/\hbar^2} r_a \right) K_{l+|k+\beta_0|+1/2} \left(\sqrt{-2mE/\hbar^2} r_b \right)}{\frac{2m}{\hbar a} I_{l+|k+\beta_0|+1/2} \left(\sqrt{-2mE/\hbar^2} a \right) K_{l+|k+\beta_0|+1/2} \left(\sqrt{-2mE/\hbar^2} a \right) - \frac{\hbar}{\alpha a^2}}. \quad (4.7)$$

The energy levels of bound states are determined by

$$\frac{\hbar^2}{2m\alpha a} = I_{l+|k+\beta_0|+1/2} \left(\sqrt{-2mE_n/\hbar^2} a \right) K_{l+|k+\beta_0|+1/2} \left(\sqrt{-2mE_n/\hbar^2} a \right). \quad (4.8)$$

A similar analysis of two dimensional case gives the upper bound of eigenvalue for the $(l+|k+\beta_0|+1/2) \rightarrow \infty$

$$\frac{\hbar^2}{m\alpha a} \approx \left(l + |k + \beta_0| + 1/2 + \frac{2m|E_n|a^2}{\hbar^2} \right)^{-1/2} < \frac{1}{l + |k + \beta_0| + 1/2}, \quad (4.9)$$

i.e.,

$$(l + |k + \beta_0| + 1/2) < \frac{m\alpha a}{\hbar^2}. \quad (4.10)$$

It is easily to see that the number of bound states of three-dimensional case is more than the two-dimensional one for an extra degree of freedom of quantum number. These results give a reasonable ground in illustrating the dependence of the electron wave functions on the vector potential of the A-B magnetic fields while electron penetrates the spherical cylinder.

ACKNOWLEDGMENTS

The work is supported by the National Science Council of Taiwan under contract number NSC88-2811-M-009-0015.

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